FSTC-HT-23-1551-71

AD735000

US ARMY FOREIGN SCIENCE AND TECHNOLOGY CENTER

THE STABILITY OF THE DISPERSION METHOD WHEN DIFFERENCE METHODS ARE USED FOR THE HEAT CONDUCTION AND WAVE EQUATIONS

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Subject Country: USSR

This document is a rendition of the original foreign text without any analytical or editorial comment.

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)
Foreign Science and Technology Center
US Army Materiel Command
Department of the Army

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP

3. REPORT TITLE
The Stability of the Dispersion Method when Difference Methods are used for the Heat Conduction and Wave Equations.

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Translation

5. AUTHOR(S) (First name, middle initial, last name)
E. C. Nikolayev

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 9, 1971	5	N/A

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

c. T702301 2301

d. Requester : Frankford Arsenal

9a. ORIGINATOR'S REPORT NUMBER(S)
FSTC-HT-23-1551-71

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT
Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY
US Army Foreign Science and Technology Center

13. ABSTRACT

The author proves the stability of the dispersion method on the upper layer, making the assumption that certain methods approximating the first boundary value problem of the heat conduction and wave equations are stable with respect to the initial conditions.

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

UNCLASSIFIED
Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Mathematic Physics Mathematic Matrix Dispersion Equation Boundary Value Problem Wave Equation Equation of State Country Code: UR Subject Code: 12						

UNCLASSIFIED
Security Classification

TECHNICAL TRANSLATION

FSTC-HT-23-1551-71

ENGLISH TITLE: The Stability of the Dispersion Method when Difference Methods are Used for the Heat Conduction and Wave Equations.

FOREIGN TITLE: Ob ustoichivosti metoda progonki pri realizatsii raznostnykh skhem dlya uravnenii teploprovodnosti i kolebanii

AUTHOR: E.C. Nikolayev

SOURCE: Journal for Numerical Analysis and Mathematical Physics, Vol. 10, No. 2, March-April, 1970

Translated for FSTC by Leo Kanner Associates, Redwood City, Ca. (Susskind)

NOTICE

It is proven in this note, that if weighted measures approximating the first boundary value problem for the heat conduction and wave equations are stable with respect to the initial conditions, the dispersion method which is used to find the solution on the upper layer is stable.

1. We will consider a weighted method, which approximates the first boundary value problem for a one dimensional heat conduction equation:

$$y_t + A(\sigma y^{j+1} + (1-\sigma) y^j) = \varphi, \quad x \in (0,1), \quad t \geqslant 0,$$
$$y(0, t) = \nu_1(t), \quad y(1, t) = \nu_2(t), \quad y(x, 0) = u_0(x), \tag{1}$$

and a symmetric method for the wave equation:

$$y_{\bar{t}t} + A(\sigma y^{j+1} + (1-2\sigma) y^j + \sigma y^{j-1}) = \varphi, \quad x \in (0,1), \quad t \geqslant \tau,$$
$$y(0,t) = \nu_1(t), \quad y(1,t) = \nu_2(t), \quad y(x,0) = u_0(x), \quad y(x,\tau) = u_1(x). \tag{2}$$

Here

$$Ay = -y_{\bar{x}x} \equiv \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \quad y_t = \frac{y^{j+1} - y^j}{\tau}, \quad y_{\bar{t}t} = \frac{y^{j+1} - 2y^j + y^{j-1}}{\tau^2},$$

h and τ are steps in space and time. The stability conditions for methods (1), (2) have, because of the initial conditions, the form (see, for example, [1], pp. 567, 582-583)

$$\sigma \geqslant \frac{1}{2^k} - \frac{h^2}{4\tau^k}. \tag{3}$$

Here $k = 1, 2$ for methods (1), (2) respectively. The problem of finding solutions to (1), (2) on the upper layer reduces to solving the first boundary value problem for an ordinary three point difference equation

$$-y_{\bar{x}x} + \frac{1}{\sigma\tau^k} y = f, \quad x \in (0,1), \tag{4}$$
$$y(0) = \nu_1, \quad y(1) = \nu_2.$$

Efficient methods for multi-dimensional nonstationary equations and iterative methods used to solve stationary equations also lead to a problem of type (4).

An efficient method for solving problem (4) is the dispersion method (see [2], pp. 281-293), whose formulas, in the given case, have the form

$$\begin{aligned} &y_i = \alpha_{i+1} y_{i+1} + \beta_{i+1}, && y_N = \nu_2, && i = 0, 1, \ldots, N-1; \\ &\alpha_{i+1} = \frac{1}{2 + h^2/\sigma\tau^2 - \alpha_i}, && \alpha_1 = 0, && i = 1, 2, \ldots, N-1; \\ &\beta_{i+1} = (\beta_i + h^2 f_i)\alpha_{i+1}, && \beta_1 = \nu_1, && i = 1, 2, \ldots, N-1. \end{aligned} \tag{5}$$

The dispersion method is stable if the condition $|\alpha_i| \leq 1$ is satisfied. However, a well-known sufficient condition for the stability of the dispersion (see [3], pp. 386-394)

$$\sigma \geq 0 \tag{6}$$

narrows down the initial class of methods for which only condition (3) holds. Thus an important class of methods with a higher order of precision with c $\sigma = 0.5 - h^2/12\tau$ for method (1), and $\sigma = \sigma_0 - h^2/12\tau^2$ $(\sigma_0 \geq 0.25 - h^2/6\tau^2)$ for method (2), which are absolutely stable with respect to the initial conditions, may not satisfy condition (6). More general than (6), the stability condition of the dispersion method is given by

Theorem 1. If methods (1), (2) are stable with respect to the initial conditions, then the dispersion method for finding the solution on the upper layer is stable.

Before we prove Theorem 1., we will consider some lemmas. We will say that the dispersion method can be applied to solve the three point difference equation if for the dispersion coefficients the conditions $|\alpha_i| \leq M_1$, $|\beta_i| \leq M_2$, hold where M_1, M_2 are some constants.

We will consider the problem

$$\begin{gathered} \Lambda y = -(a y_{\bar{x}})_x + d(x) y = f(x), \quad x \in (0, 1), \\ y(0) = \nu_1, \quad y(1) = \nu_2, \quad a(x) \geq c_1 > 0. \end{gathered} \tag{7}$$

On the set S of the grid functions v, which vanish at the end points of the interval [0,1], the tri-diagonal Jacobi matrix

$$R_{N-1} = \begin{Vmatrix} C_1 & -A_2 & 0 & \ldots & 0 & 0 & 0 \\ -A_2 & C_2 & -A_3 & \ldots & 0 & 0 & 0 \\ 0 & -A_3 & C_3 & \ldots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & -A_{N-2} & C_{N-2} & -A_{N-1} \\ 0 & 0 & 0 & \ldots & 0 & -A_{N-1} & C_{N-1} \end{Vmatrix},$$

corresponds to the operator Λ where $A_i = a_i/h^2$, $C_i = A_i + A_{i+1} + d_i$. It is easily seen that the corner minors $[R_k]$ are related by the recurrence relation

$$|R_{k+1}| = C_{k+1}|R_k| - A_{k+1}^2 |R_{k-1}|, \tag{8}$$

which holds for all k = 0, 1, 2, ...,,if we set $[R_0] = 1$, $[R_{-1}] = 0$.

Lemma 1. The solution to problem

$$\Lambda v = 0, \qquad x \in (0, 1),$$
$$v(0) = 0, \qquad v(1) = h\prod_{\alpha=1}^{N} A_\alpha^{-1}|R_{N-1}| \tag{9}$$

is positive if Λ is a positive definite operator on S, and has alternating signs if Λ is negative definite.

Since Λ has fixed sign, the solution to problem (9) exists and is unique. Using (8), we find it can be written in the form

$$v_i = h\prod_{\alpha=1}^{i} A_\alpha^{-1}|R_{i-1}|.$$

By the well-known Sylvester criterion (see [4], pp. 276-279), the positive definiteness of Λ implies, that all $|R_k| > 0$, and negative definiteness implies that the sequence $\{|R_k|\}$ has alternating signs. Since all A_α are strictly positive the lemma is proved.

Theorem 2. The dispersion method is applicable to problem (7), if the operator Λ has fixed sign on the set S. Proof: The dispersion formulas for problem (7) have the form

$$y_i = \alpha_{i+1}y_{i+1} + \beta_{i+1}, \qquad y_N = \nu_2, \qquad i = 0, 1, \ldots, N-1;$$
$$\alpha_{i+1} = \frac{A_{i+1}}{C_i - A_i\alpha_i}, \qquad \alpha_1 = 0, \qquad i = 1, 2, \ldots, N-1;$$
$$\beta_{i+1} = \frac{A_i\beta_i + f_i}{A_{i+1}}\alpha_{i+1}, \qquad \beta_1 = \nu_1, \qquad i = 1, 2, \ldots, N-1. \tag{10}$$

Therefore, the dispersion method is applicable if $C_i - A_i\alpha_i \neq 0$ for $1 \leq i \leq N-1$ We introduce the grid-function v_i as follows.

$$v_i = \alpha_{i+1}v_{i+1}, \qquad i = 0, 1, \ldots, N-2,$$
$$v_N = h\prod_{\alpha=1}^{N} A_\alpha^{-1}|R_{N-1}|.$$

Using formulas (10), we see that v_i satisfies problem (9). Assume that the conditions of the theorem are satisfied; then by Lemma 1 the solution of problem (9) is bounded and does not vanish at any point of the grid. Therefore the dispersion coefficient α_i together with the β_i are bounded. It is easily seen that the sign of α_i is determined by the fixed sign property of Λ. This proves the theorem.

We shall now find sufficient conditions for the fixed sign property of the operator Λ on S.

Lemma 2. If the condition $$d(x) \geqslant -8c_1 + \delta_1, \tag{11}$$

is satisfied for problem (7) then $\Lambda \geqslant \delta_1 E$, and if condition 12 is satisfied

$$d(x) \leqslant -\frac{2}{h^2}(a(x) + a(x+h)) - \delta_2, \tag{12}$$

then $\Lambda \leqslant -\delta_2 E$, where E is the identity operator and δ_1, δ_2 are positive constants.

In fact, let $y(x) \in S$. Using the simple inequality

$$(y_{\bar{x}}^2, 1] = \sum_{i=1}^{N} h\left(\frac{y_i - y_{i-1}}{h}\right)^2 > 8\,(y,y)$$

and the partial summation formula, we obtain, when (11) is satisfied

$$(\Lambda y, y) = \sum_{i=1}^{N-1} y_i \Lambda y_i h = (a y_{\bar{x}}^2, 1] + (dy, y) \geqslant ((8c_1 + d)\,y, y) \geqslant \delta_1 (y, y),$$

i.e. $\Lambda \geqslant \delta_1 E$.

Further using the Cauchy-Bunyakovski inequality and the arithmetic-geometric mean inequality, we obtain

$$2\left|\sum_{i=1}^{N-1} a_i y_i y_{i-1}\right| \leqslant \sum_{i=1}^{N-1} (a_i + a_{i+1}) y_i^2. \tag{13}$$

Suppose now that (12) is satisfied. Taking into consideration (13), we obtain

$$(\Lambda y, y) = \sum_{i=1}^{N} a_i y_{\bar{x},i}^2 h + (dy, y) = \sum_{i=1}^{N-1} \left[\frac{1}{h^2}(a_i + a_{i+1}) + d_i\right] y_i^2 h -$$
$$- \frac{2}{h^2} \sum_{i=1}^{N-1} a_i y_i y_{i-1} h < \sum_{i=1}^{N-1} \left[\frac{2}{h^2}(a_i + a_{i+1}) + d_i\right] y_i^2 h < -\delta_2 (y, y),$$

i.e. $(\Lambda(y, y) \leqslant -\delta_2 (y, y)$. This proves the lemma.

Note. If we denote by λ_1 and λ_{N-1} the minimum and maximum eigenvalues of the operator $\Lambda_0 y = -(a y_{\bar{x}})_x$, then it is clear that in lemma 2 we can require instead of (11) and (12) that the inequalities $d(x) \geqslant -\lambda_1 + \delta_1$, $d(x) \leqslant -\lambda_{N-1} - \delta_2$, be satisfied. Proceeding analogously as in (5), we can prove in this case that $\|y - y^*\| = O(\varepsilon h^{-2} \delta_k^{-1})$, where y and y* are the exact and approximate solutions to problem (7) respectively; ε is the relative rounding error resulting from arithmetic operations.

We will now determine the conditions for which the dispersion method will be stable for problem (7).

Lemma 3. If the condition

$$d(x) \geqslant 0, \tag{14}$$

is satisfied for problem (7) then $0 \leqslant \alpha_i < 1$; ; if condition 15 is satisfied

$$d(x) \geqslant -\frac{2}{h^2}(a(x)+a(x+h)), \tag{15}$$

then $-1 < \alpha_i \leqslant 0$, and the dispersion method is stable.

In fact, condition (14) implies that $C_i \geqslant A_i + A_{i+1}$, and condition (15) implies that $C_i \leqslant -(A_i + A_{i+1})$. From here, using $\alpha_1 = 0$ and the estimates

$$\alpha_{i+1} = \frac{A_{i+1}}{C_i - A_i\alpha_i} \leqslant \frac{1}{1+(A_i/A_{i+1})(1-\alpha_i)}$$ when (14) is satisfied and

$$\alpha_{i+1} \geqslant -\frac{A_{i+1}}{A_i(1+\alpha_i)+A_{i+1}} = -\frac{1}{1+(A_i/A_{i+1})(1+\alpha_i)}$$ when (15) is satisfied,

we establish the validity of the lemma by induction.

3. We will now prove Theorem 3. For problem (4) the conditions of lemma 3 relating to the stability of the dispersion method take on the form

$$\frac{1}{\sigma\tau^k} \geqslant 0, \tag{14'}$$

$$\frac{1}{\sigma\tau^k} \leqslant -\frac{4}{h^2}. \tag{15'}$$

Assume that conditions (3) are satisfied, i.e. the initial methods are stable with respect to the initial conditions. If, in addition, $\sigma > 0$, then (14′) is satisfied and the dispersion method is stable. If it turns out that $\sigma < 0$, then from (3) we have

$$\frac{4\tau^k}{2^k h^2} < 1 \text{ и } \frac{1}{\sigma\tau^k} \leqslant \frac{1}{\tau^k(1/2^k - h^2/4\tau^k)} = -\frac{4}{h^2}\frac{1}{1-4\tau^k/2^k h^2} \leqslant -\frac{4}{h^2}.$$

In this case (15′) is satisfied and the dispersion method is stable. This proves the theorem.

It is the author's pleasure to express his gratitude to A.A. Samarskom, who drew his attention to the problem studied above.
Submitted for publication 1/27/69

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